Classical nonintegrability of a quantum chaotic $SU(3)$ Hamiltonian system

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Abstract

We prove nonintegrability of a model Hamiltonian system defined on the Lie algebra $\mathfrak{su}_3$ suitable for investigation of connections between classical and quantum characteristics of chaos.

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1 Introduction

The problem of distinguishing on the quantum level between integrability and chaos of classical systems is a recurrent, if not the principal, topic of the quantum chaos theory. The usual definitions or signatures of the chaotic motion based on such phase-space notions like Lyapunov exponents or various degrees of ergodicity lack sense on the quantum level due to the absence of the proper notion of the phase space in quantum mechanics. Instead, in search of criteria of quantum chaos one should resort to purely quantum characteristics of a system, like e.g. its spectral properties.

A quarter of a century ago, Bohigas and coworkers [1] proposed a characterization of quantum chaos based on the statistical theory of spectra. According to their hypothesis most quantum systems whose classical limit is chaotic display universal spectral fluctuations determined by the Random Matrix Theory (RMT) [2]. Generic classically nonintegrable systems exhibit thus level repulsion i.e. the

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probability of finding two adjacent energy levels tends to zero with the difference between their energies.

A vast numerical and experimental evidence [3, 4] in favor of this hypothesis was collected during last twenty years. At least two general strategies providing theoretical arguments supporting the Bohigas-Giannoni-Schmidt conjecture were developed. The first one employs statistical mechanics of a fictitious gas of eigenvalues undergoing parametric dynamics, where the role of time is played by a control parameter controlling the transition from an integrable to a chaotic system [3, 4]. The other approach takes its roots in the semiclassical quantization *via* classical periodic orbits pioneered by Gutzwiller [5] and extends ideas from the theory of disordered systems to dynamical systems [6]. In both approach, at some stage one invokes some statistical hypothesis which justifies ascribing statistically inferred properties to an individual system which is the object of actual numerical or experimental examinations. On the other hand the proofs of nonintegrability on the classical level are usually based solely on numerical investigations. We do not know of any example of a model system for which there exist analytical proofs of the classical nonintegrability on one side and of the repulsion between quantum levels on the other. The aim of the presented investigations is to provide such a model. In the present paper we will concentrate on the classical side of the problem and show its classical nonintegrability.

In a series of papers [7, 8, 9] one of us proposed a class of models taking their origins in atomic physics and quantum optics in which dynamical variables were elements of a compact semisimple Lie algebra in some particular irreducible representation. The classical limit was attained by going with the dimension of the representation to infinity. For Lie algebras with the rank larger that one (e.g. for $\mathfrak{su}_3$ algebra) there are more than one, 'natural' ways of performing this limiting procedure. In the effect there are several inequivalent classical corresponding classical systems, differing e.g. by the dimensionality of the classical phase space. The interplay between the number of degrees of freedom and the dimensionality of the space is crucial for the (non-)integrability of a classical Hamiltonian system. A particular quantum system can be classically integrable or not, depending on the way the classical limit is approached. The story becomes interesting if, basing on the above observation, one can say something about purely quantum features (e.g. spectral properties) of the quantum system in question. The affirmative reply based on numerical investigation of spectra and classical characteristics of chaos, was given in [8].

In the present paper we examine again the above models in order to prove in an analytical, rather than numerical manner the nonintegrability. Admittedly it is a rather minimalistic goal. Nobody claims that mere nonintegrability of the classical system is sufficient for the repulsion of the quantum energy levels, we believe that the Bohigas-Giannoni-Schmidt conjecture gives correct predictions only for 'sufficiently chaotic' systems. Nevertheless the nonintegrability is a necessary prerequisite and we decide to pursue such a modest goal of showing it for the considered system.

2 Nonintegrability of Hamiltonian systems

In this section we present a brief introduction to the Morales-Ramis theory of non-integrability of Hamiltonian systems [10, 11]. The aim is to give some basic definitions and theorems (without proofs) and to present general scheme which should be followed when one wants to prove non-integrability. The whole concept is based on the differential Galois theory (here good references are [12, 13, 14]), so we also give some key ideas of it.

2.1 Differential Galois theory

Differential Galois theory (a.k.a. Picard-Vessiot theory) is an analogue of the classical Galois theory (which deals with algebraic equations), for linear differential equations. The main object of the theory is here a differential field K i.e. an algebraic field equipped with a differentiation, i.e. a linear mapping $' : K \to K$ satisfying the Leibniz rule, $(fg)' = f'g + fg'$.

The object of our interest is a linear differential equation:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0, \tag{1}$$

where the coefficients a_i are elements of K (it is good to think $K = C(x)$, the field of rational functions). The natural question which arises in the differential Galois theory is: can one solve the equation (1) using some "elementary" operations? Of course usually the solution of (1) is not contained in the field K and we have to extend K to a larger one. The smallest extension of K which contains all solutions of (1) is called the Picard-Vessiot extension. It exists always when the field of constants of K is algebraically closed [12].

To be more precise we have to define what we mean by "elementary" operations.

Definition 1 *Let K be a differential field. We say that the equation (1) is solvable in the Liouville functions category when the Picard-Vessiot extension of K can be obtained from K in a finite number of steps each one being an extension of K by adding:*

1. *α_i which is algebraic over K,*
2. *α_i which is such that $\alpha_i' \in K$,*
3. *α_i which is such that $\frac{\alpha_i'}{\alpha_i} \in K$.*

The steps 1., 2., and 3. correspond to the operations of taking roots of polynomial equations, integration, and taking the exponent of an integral — the natural operations when one solves a differential equation. The last important definition is that of the differential Galois group:

Definition 2 *Let $L \supset K$ be a Picard-Vessiot extension of K for the equation (1). The differential Galois group $Gal(L \supset K)$ of the extension $L \supset K$ is the group of all differential automorphisms (algebraic automorphisms that commute with the differentiation) of L which are identity on K.*

Let y_1 be a solution of (1) and $\sigma \in Gal(L \supset K)$. Then:

$$\sigma(y_1^{(n)} + a_{n-1}y_1^{(n-1)} + \ldots + a_1 y_1' + a_0 y_1) = \sigma(0) = 0 \tag{2}$$

Using properties of σ, namely the fact that it is a differential automorphism, we obtain:

$$\sigma(y_1)^{(n)} + a_{n-1}\sigma(y_1)^{(n-1)} + \ldots + a_1\sigma(y_1)' + a_0\sigma(y_1) = 0 \tag{3}$$

The last equality says that every element of $Gal(L \supset K)$ gives a solution of (1) when acting on a solution of (1). Let $\{y_1, y_2, \ldots, y_n\}$ be the fundamental set of solutions of (1). This means that any solution of (1) has the form $y = \sum_{i=1}^{n} \alpha_i y_i$ where α_i are some constants, and

$$\sigma(y) = \sigma\left(\sum_{i=1}^{n} \alpha_i y_i\right) = \sum_{i=1}^{n} \alpha_i \sigma(y_i). \tag{4}$$

From the last equality we see that any $\sigma \in Gal(L \supset K)$ is completely determined by its action on solutions of (1), i.e.

$$\sigma(y_i) = \sum_{i=1}^{n} a_{ij} y_j \tag{5}$$

where a_i are constants. As a result we can represent the differential Galois group as a subgroup of $GL(n, C_K)$ where C_K - the field of constants of K. In fact this is an algebraic subgroup of $GL(n, C_K)$ [15], so in particular a Lie group. Now we can formulate the main theorem [12]

Theorem 1 *The differential equation (1) is solvable in the Liouville category if and only if the corresponding differential Galois group is solvable.*

Thanks to the fact that $Gal(L \supset K)$ is a Lie group it is enough to check whether the Lie algebra of $Gal(L \supset K)$ is solvable.

2.2 Morales-Ramis theory - the general scheme

The Morales-Ramis theory is a powerful tool for checking (non-)integrability of Hamiltonian systems. By an integrable Hamiltonian we understand here one which admits enough number of functionally independent, involutive with respect to the Poisson bracket integrals of motion (this number should be equal to the number of degrees of freedom of our system). Establishing the nonintegrability of a Hamiltonian system is thus equivalent to proving the nonexistence of the appropriate number of integrals of motion.

Let us shortly outline the logic of such proofs. For a general system of nonlinear differential equations a direct count of the number of integrals of motion is difficult - in principle there are no known methods of achieving the goal. It is, however, clear that integrability of a nonlinear system should be, in some way, inherited by its linearized version. Now, for linear equations we have powerful methods based on the differential Galois theory outlined above which can be used to relate the (non)-integrability to properties of the differential Galois group. Reversing now the argument we see thus that if by analyzing the differential Galois group we are able to establish nonintegrability of the linearization, we will prove the nonintegrability of the full, nonlinear system of equations. An additional bonus is provided by the Hamiltonian character of the system which simplifies the structure of the Galois group of its linearization.

To make the above outlined idea precise and workable we need some concepts and facts. Let (M, ω) be a symplectic manifold, i.e. ω is a closed ($d\omega = 0$) and non-degenerate two-form on M. The Hamilton equations corresponding to a Hamilton function H have the form

$$\dot{x} = X_H(x), \tag{6}$$

with X_H defined *via* $\imath_{X_H}\omega := \omega(X_H, \cdot) = dH$. A linearization of (6) is now achieved by defining the variational equation (VE) along a particular non-equilibrium integral curve $\Gamma : x = \phi(t)$ of (6). By considering a solution of (6) in the form $x' = \phi(t) + \chi(t)$ and retaining only the linear terms we obtain the familiar result:

$$\dot{\chi} = X_H'(\phi(t))\chi. \tag{7}$$

For further considerations concerning additional possible simplifications and reductions of (7) it is worth treating the derivation of it on a slightly more formal level. We first observe that we can restrict the tangent bundle TM to Γ obtaining the vector bundle $TM|_\Gamma$ over Γ. We define operator D on $TM|_\Gamma$ to be the Lie derivative $\mathcal{L}_{X_H}$ restricted to $TM|_\Gamma$. More precisely to compute $\mathcal{L}_{X_H}Y$ we extend Y to $\tilde{Y}$ on a neighborhood of Γ, compute $\mathcal{L}_{X_H}\tilde{Y}$, and restrict the result to Γ. Operator D inherits the properties of the Lie derivative, in particular $D(fY) = f'Y + fD(Y)$. The variational equation along Γ is simply $DY = 0$. It can be easily checked that choosing a suitable basis in $TM|_\Gamma$ it can be written in the form (7).

Now we want to explore the above mentioned idea that if the system (6) is integrable, then the system (7) is integrable as well. To this end we use the Ziglin lemma [11] stating that with every first integral f of the system (6) we can associate a first integral f^o of (7). Moreover if $f_1, f_2, \ldots, f_k$ are involutive, functionally independent first integrals of (6) then the corresponding functions $f_1^o, \ldots, f_k^o$ are involutive, functionally independent first integrals of (7). The Morales-Ramis' idea was to investigate which restrictions on the differential Galois group of the variational equation are imposed by the complete integrability of the system (6). It turns out [10, 11] that:

Theorem 2 *Assume that a complex analytic Hamiltonian system is integrable in the meromorphic function category, then the identity component of the differential Galois group of the corresponding variational equation is abelian.*

The complete integrability indeed imposes a very strong condition on the differential Galois group of VE. To use this theorem effectively we notice that if we know k involutive integrals of motion of the system (6) we can reduce the dimensionality of the system – this is a familiar procedure known from standard classical mechanics. More precisely the involutive integrals of motion determine k commuting Hamiltonian vector fields which, in turn, define an isotropic[1] subbundle F of TM. We can now perform a symplectic reduction. We have a well defined symplectic form on $F^{\perp_\omega}/F = F^N$ and, further, we can restrict the operator D to F^N because $DY \in F^{\perp_\omega}$ if $Y \in F^{\perp_\omega}$. After the reduction we obtain from (7) a $2(n-k)$ - dimensional system called the normal variational equation (NVE) [10, 11]. It is proved in [10] that if the differential Galois group of VE is abelian then this also true for the differential Galois group of NVE. Moreover if there is only one missing first

[1]A subspace W of a symplectic space V is isotropic if and only if $W \subset W^{\perp_\omega}$, i.e. W is a subspace of its orthogonal complement in the sense of the symplectic form ω.

integral of motion then VE can be reduced to a 2-dimensional linear matrix differential equation equivalent to some second order linear differential one. Now the investigations of the Galois group can be performed in an algorithmic way. From the general theory (see Section 2.1) we know that the Galois group is a subgroup of $GL(2,\mathbb{C})$. In fact we can restrict our search to subgroups of $SL(2,\mathbb{C})$ [12]. Indeed, by a change of the dependent variable $z(t) = \exp\left(\frac{1}{2}A(t)\right)y(t)$, with $A'(t) = a_1(t)$ we can eliminate the first-derivative term from the equation $y'' + a_1 y' + a_0 y = 0$ obtaining $z'' + b_0 z = 0$ with $b_0 = a_0 - \frac{1}{4}a_0^2 - \frac{1}{2}a_0'$ without spoiling such properties of the coefficients like meromorphicity or rationality. It is a matter of a short calculation to show that the Wronskian, $W = z_1 z_2' - z_1' z_2$, of two solutions z_1, z_2 of the new equation is a constant function, a non-zero one if z_1 and z_2 are independent. Hence, from the definition of the differential Galois group we have $\sigma(W) = W$ for its arbitrary element σ. On the other hand, by a straightforward calculation, $\sigma(W) = \det(\sigma)W$, hence $\det(\sigma) = 1$.

Among four possibilities allowed [12], i.e. the differential Galois group being

1. a finite group: the tetrahedral group, the octahedral group or the icosahedral group,
2. the group of matrices conjugated to the subgroup
$$\left\{\begin{bmatrix} c & 0 \\ 0 & c^{-1}\end{bmatrix}, \begin{bmatrix} 0 & c \\ c^{-1} & 0\end{bmatrix}, \quad 0 \neq c \in \mathbb{C}\right\}$$
i.e. matrices of the form AXA^{-1}, where A is a fixed element of $SL(2,\mathbb{C})$ and X varies over the subgroup,
3. the group of triangulizable matrices, i.e. matrices conjugate to the subgroup of triangular matrices, $\left\{\begin{bmatrix} c & d \\ 0 & c^{-1}\end{bmatrix}\right\}$, and
4. the whole $SL(2,\mathbb{C})$,

only in the last case it is not solvable. A practical way of establishing the relevant case for a particular equation is provided by the Kovacic algorithm [16] which can be used to determine the differential Galois group upon analyzing poles of the coefficient b_0. Of course, if the group is not solvable then it is not abelian either, hence our system is not integrable.

3 Classical and quantum chaos of $\mathfrak{su}_3$ systems

In [8] a special class of systems having a compact phase space on the classical level and, consequently, a finite-dimensional Hilbert space in the quantum setting, was investigated. The classical limit is approached by increasing the dimension of the Hilbert space.

As an example let us consider a collection of N atoms interacting resonantly with the electromagnetic radiation. Usually, due to imposed resonance conditions, only a finite number n of energy levels of each atom is involved in the interaction. Transitions from the level $|l\rangle$ to the level $|k\rangle$ of a single atom are described by the operators $s_{kl} = |l\rangle\langle k|$, which span the defining representation of the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ in the n-dimensional Hilbert space spanned by the states of an atom. In a system of N atoms confined to a small volume in which they feel the same field

amplitude the transitions are described by the operators $S_{kl} = \sum_{\alpha=1}^{N} s_{kl}^{\alpha}$, where s_{kl}^{α} acts as s_{kl} on the levels of the α-th atom and as the identity on the rest. Clearly S_{kl} span the N-th tensor power of the defining representation of $\mathfrak{gl}_n(\mathbb{C})$ relevant for a single atom. The resulting representation is clearly reducible. Various preparation of the initial state of the whole system of atoms determine its relevant irreducible components.

Typically the number of atoms is conserved, so $\hat{N} = \sum_{i=1}^{N} S_{ii}$ is a constant of motion, and we can restrict the considerations to $\mathfrak{sl}_n(\mathbb{C})$. The observables of the considered model are constructed as polynomials in the generators. Since they have to be hermitian we finally focus our attention on the $\mathfrak{g} = \mathfrak{su}_n$ algebra and $G = SU_n$ Lie group.

The dynamics of the observables is governed by Heisenberg equations of motion generated by the Hamilton operator of the considered system. The classical limit becomes relevant when we increase the number of atoms and ask questions about such quantities like e.g. energy or polarization per one atom. Formally it consists of putting $N \to \infty$. We expect that in the limit the generators S_{kl}, after appropriate scaling (e.g. by the number of atoms), are mapped into classical functions on appropriate phase space in such a way that the Heisenberg equations of motion are mapped to classical Hamilton equations. In this way the 'Dirac quantization' procedure requesting correspondence between commutators of observables and Poisson brackets of the corresponding phase-space functions ('classical observables') is observed.

Increasing the number of atoms N results in the growing dimension of the largest irreducible component of the constructed representation. The construction of the classical phase space is achieved by the following limiting procedure. For a irreducible representation of G in a vector space V we consider its projective variant i.e. the action of the group on the projective space $\mathbb{P}(V)$ given by $g \cdot [v] = [g \cdot v]$ for $g \in G$, $v \in V$ and $[v] \in \mathbb{P}(V)$ – the ray through v. It is known [17] that the orbit of G through the point $[v] \in \mathbb{P}(V)$ corresponding to the highest weight vector, i.e. the common eigenvector of all S_{ii} annihilated by all S_{ij} with $i < j$, is endowed with a natural symplectic structure. The projective orbits through the highest weight vectors can be mapped on the orbits of the coadjoint representation of G [18] i.e. the representation of G on the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ with the corresponding symplectic structure known as the Kirillov-Kostant-Souriau form. Coadjoint orbits are thus good candidates for classical phase-spaces. Each irreducible representation and each above defined orbit is uniquely determined by the highest weight vector v or, equivalently, by the corresponding eigenvalues of S_{ii}, the number of which equals to the rank of the group. Alternatively and equivalently, to identify an irreducible representation we may use independent Casimir invariants – elements of the enveloping algebra of $\mathfrak{g}$. From the definition they commute with all elements of $\mathfrak{g}$, hence for each irreducible representations they are constant multiples of the identity operators. The values of these constants identify a representation and consequently also an orbit.

The symplectic structure on coadjoint orbits can be also obtained from a natural Poisson structure on the linear space $\mathfrak{g}^*$, so called Lie-Poisson structure (see next Section). Its symplectic leaves, i.e. manifolds on which the Poisson bracket determines a non-degenerate two-form, are exactly the orbits of the coadjoint representation. Finally thus we can identify the classical phase space with a symplectic leaf of the Lie-Poisson structure on $\mathfrak{g}^*$.

For each element of the sequence of irreducible representations with growing dimensions we obtain thus a unique well defined classical phase space. After an appropriate scaling by the volume of the orbit [19] we obtain in the limit the desired phase space of the limiting classical system, dynamics of which is connected with the quantum one *via* Dirac's correspondence. The general idea of this construction goes back to Simon [20]; see [9, 19] for the setting relevant for the present considerations.

The above presented construction of the classical limit is purely geometrical. To make it more appealing form the physical point of view let us observe that the final result can be also obtained by treating expectation values of the quantum observables as classical phase-space functions in the limit of vanishing Planck constant [7, 8]. The relevant expectation values are calculated for appropriate coherent states of the group G – these are the states 'most classical' from the point of view of uncertainty principle, hence the best approximations to the classical description of a system [21].

The symplectic leaves of the Lie-Poisson structure may have different dimensions. In the construction outlined above the dimensionality of the resulting phase space may thus depend on the chosen way through the sequence of irreducible representations. In the simplest non-trivial case of three-level atoms the algebra of observables is spanned by eight quantities, the generators of the Lie group SU_3. Irreducible representations of SU_3 are indexed by two independent quantum numbers – the weight of the highest-weight vector [22] – which determine also the dimension of the representation. In effect there exist two inequivalent ways to the classical limit resulting in a six- (in a generic case) or four- (in a degenerate case) dimensional classical phase space. This observation was a basic point of the paper [8], where it was shown that the dimensionality of the classical space determines not only integrability properties of a specific class of classical Hamilton functions, but also some statistical properties of spectra of the quantum Hamiltonians for which the classical system in question is the classical limit outlined above. Both investigations of spectra on the quantum level and integrability properties on a classical one were performed numerically. We are now in position to prove analytically the non-integrability for a concrete member of the considered class.

The Hamiltonian we consider is quadratic in the generators S_{ij},

$$\widehat{H} = 3(S_{12}^2 + S_{21}^2) + 15(S_{13}S_{32} + S_{23}S_{31}) \tag{8}$$

(see remarks in [7, 8] for the possibilities of experimental realizations). It is easy to show that $[H, Y] = 0$, where $\widehat{Y} = S_{11} + S_{22} - 2S_{33}$. The commutation relation survives the classical limit providing thus an integral of motion. As a result we obtain a classical Hamiltonian system with the Hamilton function

$$H = 3(s_{12}^2 + s_{21}^2) + 15(s_{13}s_{32} + s_{23}s_{31}), \tag{9}$$

admitting an integral of motion

$$Y = s_{11} + s_{22} - 2s_{33}, \tag{10}$$

where s_{ij} are coordinates on $\mathfrak{su}_3^*$ dual to S_{ij}.

In the next section we explain how to obtain from (9) the corresponding Hamilton equations of motion on $\mathfrak{su}_3^*$, in particular we describe the above mentioned Lie-Poisson structure. Since the classical phase space is either four or six dimensional the classical system (9) is integrable in the former and possibly non-integrable (if there are no other unknown integrals of motion) in the latter case.

4 Lie-Poisson structure on $\mathfrak{su}_3^*$

As stated in the Introduction and explained in the preceding section, we are interested in a class of systems which are obtained as the classical limits of some quantum systems with $SU(3)$ symmetry [7, 8, 9], and the classical limit of such a system can be considered as a Hamiltonian system on $\mathfrak{su}_3^*$. It is instructive to consider the problem in a slightly more general setting where $\mathfrak{su}_3$ is substituted by an arbitrary Lie algebra.

Let $(\mathfrak{g}, [\cdot,\cdot])$ be an arbitrary Lie algebra. We can equip its dual space $\mathfrak{g}^*$ with the canonical Lie-Poisson structure given by the Poisson bracket,

$$\{f,g\}(x) = \langle x, [(df)_x, (dg)_x]\rangle, \tag{11}$$

where $f, g \in C^\infty(\mathfrak{g}^*)$ and $\langle,\rangle$ denotes the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. The bracket (11) is well defined because $(df)_x : \mathfrak{g}^* \to R$ is an element of $\mathfrak{g}^{**}$ and in the case of finite dimensional vector spaces we have $\mathfrak{g}^{**} = \mathfrak{g}$. The bracket defined in this way is of course bilinear and antisymmetric. It is also a differentiation i.e. it satisfies the Leibniz rule. It is convenient to describe the Lie-Poisson structure on $\mathfrak{g}^*$ in terms of a Poisson bivector,

$$\eta = c_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \tag{12}$$

where c_{ij}^k are the structure constants of $\mathfrak{g}$ corresponding to some basis $e_1, ..., e_n$, i.e. $[e_i, e_j] = c_{ij}^k e_k$, and coordinates $x_1, ..., x_n$ are the vectors $e_1, ..., e_n$ considered as linear functions on $\mathfrak{g}^*$. To show that the bivector induced by Poisson bracket (11) is the same as the one given by Eq. (12), it is enough to check it on the linear functions. After completing this easy taks we may thus write the Hamilton equations of motion for na arbitrary function f on $\mathfrak{g}^*$,

$$\frac{df}{dt} = \{H, f\} = \eta(H, f) \tag{13}$$

The Poisson bivector η is degenerate on $\mathfrak{g}^*$ due to the existence of Casimir functions which have a vanishing Poisson bracket with any function. Thus we do not obtain directly any symplectic structure on $\mathfrak{g}^*$. On the other hand when we restrict η to its symplectic leaves determined by constant values of independent Casimir functions we end up with well defined symplectic manifolds. Indeed, η defines a morphism

$$\eta^\sharp : T\mathfrak{g}^{**} \to T\mathfrak{g}^*, \quad df|_x \mapsto X_f|_x, \tag{14}$$

where $X_f = \{f, \cdot\}$. It generates a distribution $\mathcal{D} = \bigcup_x D_x$, $x \in M$, and $D_x = image(\eta_x^\sharp)$. This distribution is involutive, (i.e. $[X, Y] \in \mathcal{D}$, for $X, Y \in \mathcal{D}$), hence from the Frobenius theorem $\mathcal{D}$ is tangent to some generalized foliation $\mathcal{F}$. The restriction of η to leaves, $\mathcal{F}_x$, $x \in M$, of the foliation $\mathcal{F}$ is a well defined, non-degenerate Poisson bivector, so it defines a symplectic structure on F_x. The symplectic leaves $\mathcal{F}_x$ are exactly the coadjoint orbits of G, and the corresponding symplectic form providing a symplectic structure is the announced Kirillov-Kostant-Souriau one [18]. Its explicit form can be easily deduced from the definition of η, It is, however, often easier to work with the Poisson structure (11) on the whole $\mathfrak{g}$ than with its restriction to leaves and treat the Casimir functions as constants motion determining by their initial values a manifold to which the motion is restricted. This is the way we will follow in our case.

We may now specify the above general considerations to the $\mathfrak{su}_3$ case of the present interest. To this end we have to chose some basis $\mathfrak{su}_3$ and find explicitly the bivector η. The basis of our choice consists of the standard Gell-Mann matrices (see e.g. [22]; we use them in slightly different order) multiplied by the imaginary unit i,

$$\begin{aligned}
e_1 &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
e_4 &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad e_5 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \quad e_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \\
e_7 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}, \quad e_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{bmatrix}.
\end{aligned} \tag{15}$$

In this basis we have $\eta = c_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} = \eta_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ with the coefficients η_{ij} given as

$$\begin{array}{llll}
\eta_{12} = 2x_3, & \eta_{25} = -x_6, & \eta_{46} = -x_1, & \eta_{78} = \sqrt{3}x_6. \\
\eta_{13} = -2x_2, & \eta_{26} = x_5, & \eta_{47} = x_2, & \\
\eta_{14} = -x_6, & \eta_{27} = -x_4, & \eta_{48} = -\sqrt{3}x_5, & \\
\eta_{15} = -x_7, & \eta_{34} = x_5, & \eta_{56} = -x_2, & \\
\eta_{16} = x_4, & \eta_{35} = -x_4, & \eta_{57} = -x_1, & \\
\eta_{17} = x_5, & \eta_{36} = -x_7, & \eta_{58} = \sqrt{3}x_4, & \\
\eta_{23} = 2x_1, & \eta_{37} = x_6, & \eta_{67} = (-x_3 + \sqrt{3}x_8), & \\
\eta_{24} = x_7, & \eta_{45} = (x_3 + \sqrt{3}x_8), & \eta_{68} = -\sqrt{3}x_7, &
\end{array}$$

The bivector η is degenerate and there are two functionally independent Casimir functions c_1, c_2, $(\eta^\sharp(dc_1) = 0 = \eta^\sharp(dc_2))$, given by $c_1 = \alpha \mathrm{tr}(X^2)$ and $c_2 = \beta \mathrm{tr}(X^3)$, where X is a generic matrix belonging to $\mathfrak{su}_3$ algebra,

$$X = \begin{pmatrix} ix_3 + i\frac{x_8}{\sqrt{3}} & x_1 + ix_2 & x_4 + ix_5 \\ -x_1 + ix_2 & -ix_3 + i\frac{x_8}{\sqrt{3}} & x_6 + ix_7 \\ -x_4 + ix_5 & -x_6 + x_7 & -2i\frac{x_8}{\sqrt{3}} \end{pmatrix},$$

and α, β are arbitrary constants. To keep the consistency with [8] where hermitian rather than antihermitian matrices were used to represent $\mathfrak{su}_3$ algebra, we choose $\alpha = -1$, $\beta = i$.

5 The non-integrability proof

As we noticed in Section 4, the $\mathfrak{su}_3^*$ Lie-Poisson structure has two Casimir functions c_1 and c_2, hence the dimension of a generic leaf is six, but there are cases for which it reduces to four [7]. It is interesting to check whether a Hamiltonian system defined on the whole $\mathfrak{su}_3^*$ which possess an additional first integral besides

the Hamilton function itself, 'placed' on leaves of different dimension is integrable or not. By 'placed' we mean that initial conditions determine the leaf on which the time evolution takes place. Of course such a Hamiltonian system is integrable on four dimensional leaves (there are two first integra ls in involution), but in the case of six dimensional leaves an additional first integral needed for integrability may lack. Our aim is to prove that for some particular polynomial Hamiltonian systems an additional first integral is indeed missing, with (9) treated as a concrete example.

Hamilton functions H on $\mathfrak{su}_3^*$ we are interested in are given as second order polynomials in the x_i coordinates given in Section 4. The particular example given by (9) takes in the new variables the form[2],

$$H = 6(x_1^2 - x_2^2 - 5x_4x_6 - 5x_5x_7). \tag{16}$$

The resulting Hamilton equations are given by:

$$\frac{dx}{dt} = \eta^\sharp(dH). \tag{17}$$

In the coordinates x_i they form a set of eight differential equations:

$$\begin{aligned}
\frac{dx_1}{dt} &= 24x_2x_3 + 30x_4^2 + 30x_5^2 - 30x_6^2 - 30x_7^2,\\
\frac{dx_1}{dt} &= 24x_1x_3,\\
\frac{dx_3}{dt} &= -48x_1x_2 + 60x_5x_6 - 60x_4x_7,\\
\frac{dx_4}{dt} &= -30x_1x_4 + 30x_2x_5 - 12x_1x_6 - 12x_2x_7 + 30x_7(x_3 + \sqrt{3}x_8),\\
\frac{dx_5}{dt} &= -30x_2x_4 - 30x_1x_5 + 12x_2x_6 - 12x_1x_7 + 30x_6(-x_3 - \sqrt{3}x_8),\\
\frac{dx_6}{dt} &= 12x_1x_4 - 12x_2x_5 + 30x_1x_6 + 30x_2x_7 + 30x_5(-x_3 + \sqrt{3}x_8),\\
\frac{dx_7}{dt} &= 12x_2x_4 + 12x_1x_5 - 30x_2x_6 + 30x_1x_7 + 30x_4(x_3 - \sqrt{3}x_8),\\
\frac{dx_8}{dt} &= 0.
\end{aligned} \tag{18}$$

The last equations reflects the fact that (10) is a constant of motion, since in the new coordinates we have simply $Y = x_8$. As we announced we are working in the full $\mathfrak{su}^*$ space, we know thus two additional integrals of motion given by the Casimir functions c_1 and c_2. In [8] it was shown that the for the classical limit of $SU(3)$ systems obtained *via* the procedure outlined in Section 3, the values of the Casimir constants of motion can be parameterized by a single number $q \in [0, 1]$,

$$\begin{aligned}
c_1 &= \frac{2}{3}(q^2 - q + 1),\\
c_2 &= \frac{1}{9}(-2q^3 + 3q^2 + 3q - 2).
\end{aligned} \tag{19}$$

It was also shown in [8] that for $q = 0$ and $q = 1$ the leaf on which the system evolves is four dimensional whereas for $q \in]0, 1[$ the leaves are six dimensional.

[2] The change of variables from s_{ij} to x_i can be easily red off from (4) *via* $X_{ij} = s_{ij}$

Fixing the values of c_1 and c_2 by choosing a particular q restricts also possible values of x_8. A possible compatible choice is

$$x_8 = \frac{\sqrt{3}}{2}(1 - 2q). \tag{20}$$

To use Morales-Ramis theory we have to find some particular, as simple as possible, non-equilibrium solution. If we put

$$x_1 = x_2 = x_5 = x_6 = x_8 = 0, \quad x_4 = x_7, \tag{21}$$

the system (18) reduces to two differential equations:

$$\begin{aligned} \frac{dx_3}{dt} &= -60x_4^2, \\ \frac{dx_4}{dt} &= 30x_3x_4. \end{aligned} \tag{22}$$

The system can be easily solved since we know two constant Casimir functions. We want to choose such values of them that the corresponding symplectic leaf is six-dimensional. For our choice (21) the Casimir functions simplify to $c_1 = 2(x_3^2 + 2x_4^2)$ and $c_2 = 0$. Observe that a choice $q = \frac{1}{2}$ which, as stated above corresponds to a six-dimensional phase space, indeed gives $c_2 = 0$ [see (19)], forcing in addition $x_8 = 0$ [see (20)]. We will prove that in the case $q = \frac{1}{2}$ the whole system (18) is not integrable in the Liouville sense.

The solution of (22) for $q = \frac{1}{2}$ is found to be

$$x_3 = \frac{1}{2}\tanh(-15t) \qquad x_4 = \frac{\sqrt{2}}{4}\sqrt{1 - \tanh(-15t)}, \tag{23}$$

It defines a particular integral curve Γ of the full system. We can now find variational equation along the obtained solution,

$$\frac{dy}{dt} = A(t)y, \tag{24}$$

where the matrix $A(t)$ is given by

$$A(t) = \begin{pmatrix} 0 & 24x_3 & 0 & 60x_4 & 0 & 0 & -60x_4 & 0 \\ 24x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -60x_4 & 0 & 0 & -60x_4 & 0 \\ -30x_4 & -12x_4 & 30x_4 & 0 & 0 & 0 & 30x_3 & 30\sqrt{3}x_4 \\ -12x_4 & -30x_4 & 0 & 0 & 0 & -30x_3 & 0 & 0 \\ 12x_4 & 30x_4 & 0 & 0 & -30x_3 & 0 & 0 & 0 \\ 30x_4 & 12x_4 & 30x_4 & 30x_3 & 0 & 0 & 0 & -30\sqrt{3}x_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The tangent space at each point x of curve Γ is D_x where $D_x = image(\eta_x^\sharp)$. It can be described effectively using Casimir functions, namely $V_x \in T_xF_x$ if and only if $dc_1(V_x) = 0$ and $dc_2(V_x) = 0$. If we denote by $V_x = \{V_x^1, \ldots, V_x^8\}$

the coefficients of V_x with respect to the basis corresponding to $x_1, \dots, x_8$, the conditions $dc_1(V_x) = 0$ and $dc_2(V_x) = 0$ take the form:

$$\begin{aligned} x_3 V_x^3 + x_4(V_x^4 + V_x^7) &= 0, \\ 3x_4^2 V_x^1 + 3x3 + x_4(V_x^4 - V_x^7) + \sqrt{3}(x_3^2 - x_4^2)V_x^8 &= 0 \end{aligned} \tag{25}$$

Using these relations we notice that V_x^3 and V_x^8 are completely determined if we know V_x^1, V_x^4, V_x^7, so we can reduce (24) to a set of six differential equations

$$\frac{d\xi}{dt} = B(t)\xi, \tag{26}$$

where the matrix $B(t)$ is given by:

$$B(t) = \begin{pmatrix} 0 & 24x_3 & 60x_4 & 0 & 0 & -60x_4 \\ 24x_3 & 0 & 0 & 0 & 0 & 0 \\ \frac{30x_4(x_3^2+2x_4^2)}{-x_3^2+x_4^2} & -12x_4 & \frac{30x_4^2(-4x_3^2+x_4^2)}{x_3^3-x_3x_4^2} & 0 & 0 & \frac{30(x_3^4+x_3^2x_4^2+x_4^4)}{x_3^3-x_3x_4^2} \\ -12x_4 & -30x_4 & 0 & 0 & -30x_3 & 0 \\ 12x_4 & 30x_4 & 0 & -30x_3 & 0 & 0 \\ 30x_4 + \frac{90x_4^3}{x_3^2-x_4^2} & 12x_4 & \frac{30(x_3^4+x_3^2x_4^2+x_4^4)}{x_3^3-x_3x_4^2} & 0 & 0 & \frac{30x_4^2(-4x_3^2+x_4^2)}{x_3^3-x_3x_4^2} \end{pmatrix}.$$

The set of differential equations (26) can be reduced to a normal variational equation (NVE) using the fact that $X_H = \eta^\sharp(dH)$ and $X_8 = \eta^\sharp(dx_8)$ are solutions of (26). To perform the reduction we need the symplectic form at each point of the curve Γ. It can be obtained by inversion of η_Γ which is the restriction of η to the curve Γ. It is possible since η is non-degenerate on Γ ($\Gamma \subset F_x$ for some x). Explicit calculations give the symplectic form ω_Γ along Γ as $\omega_\Gamma = \omega_{ij} dx_i \wedge dx_j$, where i and j belong to $\{1,2,4,5,6,7\}$ and the coefficients ω_{ij} read:

$$\begin{array}{ll} \omega_{12} = -\frac{x_3}{2(x_3^2-x_4^2)} & \omega_{27} = \frac{x_4}{2(x_4^2-x_3^2)} \\ \omega_{15} = \frac{x_4}{2(x_4^2-x_3^2)} & \omega_{45} = \frac{x_4^2-2x_3^2}{2x_3(x_3^2-x_4^2)} \\ \omega_{16} = \frac{x_4}{2(x_3^2-x_4^2)} & \omega_{46} = \frac{x_4^2}{2x_3(x_3^2-x_4^2)} \\ \omega_{24} = \frac{x_4}{2(x_3^2-x_4^2)} & \omega_{57} = \frac{-x_4^2}{2x_3(x_3^2-x_4^2)} \\ \omega_{45} = \frac{2x_3^2-x_4^2}{2x_3(x_3^2-x_4^2)} & \end{array}$$

The key point now is to find a symplectic basis including X_H and X_8, i.e. a set of six vector fields such that $\eta_\Gamma \sim X_H \wedge \tilde{X}_H + X_8 \wedge \tilde{X}_8 + X \wedge \tilde{X}$. Such a basis always exist [10], and in our case it is formed by

$$\begin{aligned} X_H &= 30\,x_3x_4\left(\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_7}\right), & \tilde{X}_H &= \frac{-1}{60\,x_4}\left(\frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6}\right), \\ X_8 &= x_4\left(\frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_6}\right), & \tilde{X}_8 &= \frac{x_3^2 - x_4^2}{2\,x_3x_4}\left(\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_7}\right), \\ X &= -2x_3\frac{\partial}{\partial x_1} + x_4\left(\frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_7}\right), & \tilde{X} &= -\frac{\partial}{\partial x_2} + \frac{x_4}{2x_3}\left(\frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_6}\right). \end{aligned} \tag{27}$$

We can express the equation (26) in the basis (27) as:

$$\frac{d\chi}{dt} = P^{-1}(B(t)P - \dot{P}) = C(t)\chi \qquad (28)$$

where $\chi = P\xi$ and P is the change of basis matrix:

$$P(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & -2x_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 30x_3x_4 & 0 & 0 & \frac{x_3^2-x_4^2}{2x_3x_4} & x_4 & 0 \\ 0 & x_4 & \frac{-1}{60x_4} & 0 & 0 & \frac{x_4}{2x_3} \\ 0 & -x_4 & \frac{-1}{60x_4} & 0 & 0 & -\frac{x_4}{2x_3} \\ 30x_3x_4 & 0 & 0 & \frac{x_4^2-x_3^2}{2x_3x_4} & -x_4 & 0 \end{pmatrix}$$

The derivative matrix $\dot{P}$ can be easily computed using (22). The final result reads as

$$C(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 30 - \frac{30x_4^2}{x_3^2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -30 + \frac{30x_4^2}{x_3^2} & 0 & 12 \\ 0 & 0 & 0 & 0 & 48x_3^2 & 0 \end{pmatrix}$$

Thus the NVE is a simple 2×2 matrix differential equation:

$$\frac{d\tilde{\chi}}{dt} = \begin{pmatrix} 0 & 12 \\ 48x_3^2 & 0 \end{pmatrix} \tilde{\chi}.$$

Writing

$$\tilde{\chi} = \begin{bmatrix} \tilde{\chi}_1 \\ \tilde{\chi}_2 \end{bmatrix} \qquad (29)$$

we find the corresponding second order differential equation,

$$\frac{d^2\tilde{\chi}_1}{dt^2} - 576x_3^2\tilde{\chi}_1 = 0. \qquad (30)$$

Making use of (23) we get:

$$\frac{d^2\tilde{\chi}_1}{dt^2} - 144\tanh^2(-15t)\tilde{\chi}_1 = 0 \qquad (31)$$

The substitution $y = \tanh(-15t)$ transforms the equation to:

$$255\left(1-y^2\right)^2\frac{d^2f}{dy^2} - 450\left(1-y^2\right)y\frac{df}{dy} - 144y^2f = 0 \qquad (32)$$

The equation is fully prepared to be treated by the Kovacic algorithm. We are not going to describe it here (see the references [16] and [10] for details). The algorithm produces a solution if an equation is integrable in the Liouville

sense, and, what is more important for us, it determines as a byproduct the differential Galois group identifying it as one among those listed at the end of Section 2.2. For our equation (32) the result is that the Galois is not solvable. It is thus not abelian and the Hamiltonian (16) is not integrable in the meromorphic function category on the chosen, six-dimensional phase space. To check the Liouville integrability in a concrete case like (32) one can also use an implementation of the Kovacic algorithm in symbolic manipulation programs, e.g. *kovacicsols* from *Maple 12* which returns a list of Liouvillian solutions if they exist and the empty set in the opposite case. The occurrence of the latter case is thus a proof of the nonintegrability of the full system.

6 Conclusions and outlook

We proved in an analytical way the non-integrability of a specific quadratic Hamilton function defined on the Lie algebra $\mathfrak{su}_3$ obtained as a classical limit of a quantum Hamiltonian. The motivation was thoroughly presented in the Introduction and Section 3, here we want to conclude that the achieved result fills, at least partially, a gap in the reasoning of [8].

We would like also to highlight some novelties of our investigation. Since the birth of the Morales-Ramis theory there has been many successful attempts to apply it to concrete physical situations [23, 24, 25, 26]. The investigated system were, usually, of the standard type with Hamilton function of the form of a sum of the kinetic and potential energies, the kinetic energy being a quadratic form in the canonical[3] momentum variables, defined in a topologically simple phase space. This is not the case for the Hamilton function treated in our paper. It is a quadratic polynomial in non-canonical variables on a compact symplectic manifold.

The presented reasoning can be applied to other symplectic leaves, other Hamilton functions on $\mathfrak{su}_3$ (or other Lie algebras). The desired result would be a classification of such Hamilton functions with respect to their integrability.

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